THE BELL SYSTEM TECHNICAL JOURNAL
Vol. 54, No. 2, February 1975
Printed in U.S.A.

Applications of Group Theory to Connecting Networks

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(Manuscript received May 30, 1974)

Group theory impinges on the combinatorial study of connecting networks in a natural way: the stages, frames, and cross-connect fields from which many existing networks are built provide simple permutations out of which desired, complex assignments are built by composition. Some of the consequences of this interpretation are explored in this paper. In the group-theoretic setting, the action and role of the stages and fields become transparent, and many questions and results regarding networks can be regarded as problems about cosets, subgroups, factorizations, etc. This approach is particularly useful for the study of rearrangeable networks made of stages of square switches; such a network is rearrangeable if and only if the symmetric group of appropriate degree can be factored into products of certain subgroups associated with the network. Or again, the original Slepian-Duguid rearrangeability theorem corresponds to factoring a symmetric group into a product of double cosets of subgroups generated by stages.

I. INTRODUCTION AND SUMMARY

Many connecting networks for telephone switching are constructed of several stages of independently acting rectangular or square switches with suitable cross-connect fields between tbe stages to allow for "grosser" transitions. The permutations or maps of inlets into outlets tbat can be realized by the network are obtained in a sequence of steps eacb corresponding to passage througb a stage or a cross-connect field. To put it in mathematical terms, the stages and cross-connect fields provide simple maps out of which desired ones can be built by successive *compositions*. This fact allows one to use tbe concepts of group theory to study questions about connecting networks. Sucb a study was initiated in a previous paper,[1] in whicb we remarked tbat it always seemed to be easier to obtain results about groups by the few available methods known for networks than vice versa. We are happy to report that this tendency has been in part reversed.[2]

In Sections II and III we describe how the actions of a stage of switching and that of a cross-connect field are to be interpreted in terms of *permutations*. Section IV contains a definition of a general notion of a "stage," and it is explained there how the cascading of successive stages of switching separated by cross-connect fields corresponds to *composition* or multiplication of permutations. In Section V, we indicate how some concepts from group theory can be used to describe the permutations achievable by successive stages of square switches with fields between them. As a result, we can in Section VI pose some questions about (permutation) groups that are relevant to the practical matter of what calls can be carried in a network made of stages of square switches.

Next, Section VII contains a study of how stages can give rise to subgroups, culminating in the result that a stage $\mathcal{S}$ is made of square switches iff $\mathcal{S}$ generates a group and $\mathcal{S}$ is complete in this sense: every crosspoint of $\mathcal{S}$ is used in some permutation that $\mathcal{S}$ can generate. The factoring of groups into products of complexes or subgroups is taken up in Section VIII. This important phenomenon first arose in a group-theoretic interpretation[1] of the rearrangeability theorem of Slepian and Duguid, and has since appeared in other studies[2] of this basic network property. Some half-dozen theorems on factoring a group into a product of complexes or subgroups are given. The special case of factoring by double cosets, exemplified by the Slepian-Duguid results, is considered in Section IX, and it is shown that only the standard "frame" cross-connect field used in that result will give a rearrangeable network when identical square switches are used in a stage.

II. STAGES AND CROSS-CONNECT FIELDS

Two examples of connecting networks are shown in Figs. 1 and 2. They illustrate how networks can be built of stages of (usually square) switches joined by a *link pattern* or *cross-connect field*.* These fields are responsible for the distributive characteristics of the network. They afford an inlet many ways of reaching other switches and so many outlets. The examples have the property that the number of inlets, the number of links in a cross-connect field, and the number of outlets are all the same number. We shall restrict our attention to networks with this property, built by alternating stages with link patterns.

* The word "field" in this usage is borrowed from the domain of switching engineering, and has no algebraic significance. Such a field is, of course, usually tantamount to a permutation.

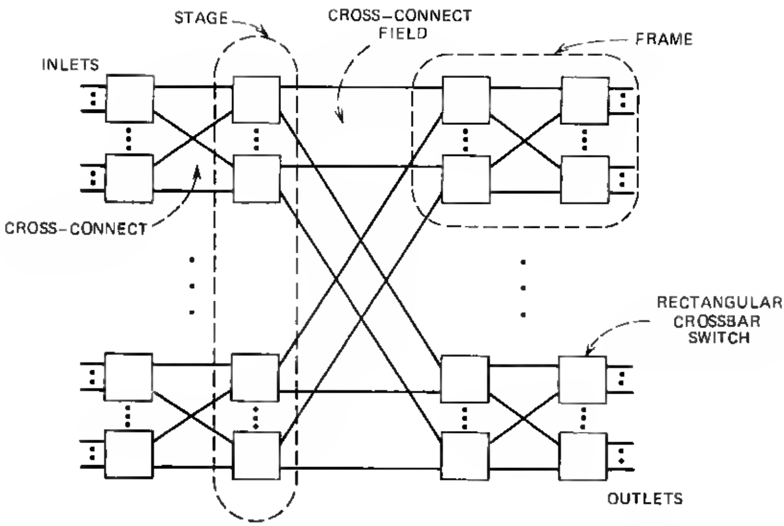

Fig. 1—Network showing stages, frames, and fields.

III. INTERPRETATION IN TERMS OF PERMUTATIONS

Suppose that the inlets and the outlets are both numbered in some arbitrary way from 1 to N. Then it is clear that each link-pattern, and each permitted way of closing N crosspoints in a stage, can be viewed abstractly as a permutation on $\{1, \cdots, N\}$. Here, "permitted," of course, means that no inlet to a stage is connected to more than one outlet, nor is any outlet connected to more than one inlet. Both the examples have the property that any maximal state, i.e., one in which no additional calls can be completed, has exactly N calls in progress; such a state realizes a permutation that is a product of the permutations represented by the link-patterns and the switch settings in the stages.

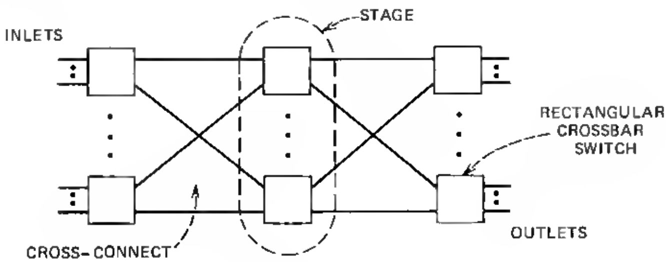

Fig. 2—Three-stage network.

IV. STAGES AND PRODUCTS

It will be convenient to adopt a generalized notion of a "stage" of a switching network. This generalization is based on the view that a stage is essentially a two-sided connecting network in which every call passes through one crosspoint. By a stage of switching we shall mean a connecting network constructed thus: with I the set of inlets and Ω the set of outlets, we choose an arbitrary subset $\mathcal{S}$ of $I \times \Omega$, and we place a crosspoint between all and only those inlets $u \in I$ and outlets $v \in \Omega$ such that $(u, v) \in \mathcal{S}$, and we speak of $\mathcal{S}$ itself as the stage. Logically, $\mathcal{S}$ is a relation indicating between what inlets and outlets there are crosspoints; it thus specifies the structure of the stage in the sense of Ref. 2. Thus,

Definition 1: A stage is a subset of $I \times \Omega$.

This terminology is an extension of the usual one, according to which, for example, a column of switches in Fig. 1 or 2 forms a stage, and the network consists of four or three stages joined by three or two cross-connect fields.

Definition 2: A substage $\mathcal{S}'$ of a stage $\mathcal{S}$ is a subset of $\mathcal{S}$.

In view of the discussion in Section III, we henceforth identify $I = \Omega = \{1, \cdots, N\}$.

Definition 3: A stage $\mathcal{S}$ is made of square switches iff there is a partition Π of $\{1, \cdots, N\}$ such that

$$\mathcal{S} = \bigcup_{A \in \Pi} (A \times A).$$

Evidently, $\mathcal{S}$ is made of square switches iff it is an equivalence relation. It is easily seen why all the stages illustrated in Figs. 1 and 2 are made of square switches as stated in Definition 1. Consider a stage $\mathcal{S}$ that has N inlets and N outlets. Evidently, such a stage provides ways of connecting some of the inlets simultaneously to some of the outlets. If the stage contains enough cross-points, it can be used to connect every inlet to an outlet, with no inlet connected to more than one outlet and vice versa; such a switch setting corresponds to a permutation on $\{1, \cdots, N\}$. This circumstance motivates the following definition.

Definition 4: A stage $\mathcal{S}$ generates the permutation π on $\{1, \cdots, N\}$ if there is a setting of N crosspoints of $\mathcal{S}$ which connects i to $\pi(i)$, $i = 1, \cdots, N$, that is, if $[i, \pi(i)] \in \mathcal{S}$ for $i = 1, \cdots, N$, or most simply if $\pi \subseteq \mathcal{S}$.

Definition 5: $P(\mathcal{S})$ is the set of permutations generated by $\mathcal{S}$.

Notice that $P(\mathcal{S})$ may be empty, and that for many π, $\mathcal{S}$ may generate* various submaps of π without generating π itself.

Definition 6: A network with N inlets and N outlets generates a permutation π if there is a setting of the crosspoints of the network which connects, by mutually disjoint paths, each inlet to a unique outlet such that i is connected to $\pi(i)$, $i = 1, \cdots, N$.

Multiplication of permutations is defined in the usual way by *composition.* Thus, if π_1 and π_2 are permutations, then $\pi_2\pi_1$ is tbe permutation defined by

$$\pi_2\pi_1(i) = \pi_2[\pi_1(i)] \quad i = 1, \cdots, N.$$

If two stages $\mathcal{S}_1$ and $\mathcal{S}_2$ are connected by a link pattern corresponding to a permutation π, then togetber they generate the permutations of tbe form

$$\varphi_2\pi\varphi_1 \quad \varphi_i \in P(\mathcal{S}_i), \quad i = 1 \text{ and } 2.$$

V. CONNECTION WITH GROUP THEORY

We adopt some concepts and notations from group theory to simplify the presentation. If G is a group, it is customary (although now a little old-fashioned) to speak of a subset $K \subseteq G$ as a *complex.* If $x \in G$, then xK denotes the set of products xy witb $y \in K$, and $Kx = \{yx: y \in K\}$. Similarly, for complexes K_1 and K_2, the product K_1K_2 is the set of products yz with $y \in K_1$ and $z \in K_2$.

If a network consists of two stages $\mathcal{S}_1$ and $\mathcal{S}_2$ joined by a cross-connect field corresponding to a permutation π, then it generates exactly the permutations in the complex

$$P(\mathcal{S}_2)\pi P(\mathcal{S}_1).$$

Similarly, a network ν of s stages $\mathcal{S}_i$, $i = 1, \cdots, s$, with a cross-connect field π_i between $\mathcal{S}_i$ and $\mathcal{S}_{i+1}$, $i = 1, \cdots, s - 1$, generates the complex

$$P(\nu) = P(\mathcal{S}_s)\pi_{s-1}\cdots P(\mathcal{S}_2)\pi_1 P(\mathcal{S}_1). \tag{1}$$

This complex completely describes the maximal assignments realizable by a network built of stages joined by link-patterns, all of N inlets and N outlets. To ask what simultaneous calls the network can carry is to ask what permutations of the full symmetric group S_N on $\{1, \cdots, N\}$ belong to the complex. This is a question of group theory that can in some cases be answered by its methods.[1,2] It is now possible to formulate a group-theory approacb to tbe analysis and syntbesis of

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* To use an obvious extension of the terminology of Definition 4.

connecting networks made of stages, for the factors $P(\mathcal{S}_i)\pi_i$, $i = 1, \cdots, b-1$ and $P(\mathcal{S}_b)$ occurring in (1) are themselves complexes.

VI. QUESTIONS

With this interpretation of the combinatorial power of a network in mind, we can immediately pose several problems of (permutation) group theory that shed light on the practical question, What calls can be carried in a network?

(*i*) What products of complexes are groups?
(*ii*) What groups can be generated by stages?
(*iii*) When can the whole symmetric group S_N be factored into a product of complexes corresponding to stages joined by cross-connect fields? In other words, When does such a product correspond to a *rearrangeable*[1] network?
(*iv*) What relationships and trade-offs exist between the stages and cross-connect fields chosen to build a network and the assignments it can realize?

Aspects of the first three questions will be taken up in the following sections; the fourth is discussed in Ref. 2.

VII. GENERATION OF GROUPS BY STAGES

In studies of rearrangeability of networks, questions have arisen as to (*i*) when the set $P(\mathcal{S})$ of permutations generated by a stage forms a group and (*ii*) what groups can be got in this way. Only a partial answer has been given.[1] In cases of practical importance, such as those in Figs. 1 and 2, the stages are made of square switches. Clearly, such a stage is capable of effecting or generating only a special class of permutations: for each switch there are numbers m and n with $m < n$ such that the switch can perform all $(n - m + 1)!$ permutations of the numbers k in the range $m \leq k \leq n$ among themselves. Since no inlet or outlet is on more than one switch, the permutations generated by a stage form a *subgroup* of the symmetric group S_N of *all* permutations on $\{1, \cdots, N\}$. This subgroup is isomorphic to the direct product $\Pi_i S_{n_i}$, where n_i are the switch sizes; i.e., the subgroup has a property which might be described intuitively by saying that there exist sets on which the subgroup elements can "mix strongly," but which they keep separate. Group theory has some terminology for this situation, and we specialize it as follows.

A group G of permutations is called *imprimitive*[3] iff the objects acted on by the permutations of G can be partitioned into mutually disjoint sets, called the *sets of imprimitivity*, such that every $\pi \in G$

either permutes the elements of a set among themselves, or else carries that set onto another. That is, there is a partition Π of the set X acted upon such that $\pi \in G$ and $A \in \Pi$ imply $\pi(A) \in \Pi$. We shall specialize this terminology as follows:

Definition 7: G is called strictly imprimitive iff it is imprimitive and the sets of imprimitivity are carried onto themselves by elements of G, i.e., iff there is a partition Π of X such that $A \in \Pi$ implies $\pi(A) = A$ for all $\pi \in G$, so that $\pi \in G$ are nonmixing on Π.

Remark 1: Let M be a complex, i.e., a set of permutations. Define a stage $\mathcal{S}$ by

$$\mathcal{S} = \{(i, j): \pi(i) = j \text{ for some } \pi \in M\}.$$

Then $P(\mathcal{S}) \supseteq M$, and no smaller stage has this property.

Remark 2: If $\mathcal{S}$ is made of square switches, then $P(\mathcal{S})$ is a strictly imprimitive group. For $(i, i) \in \mathcal{S}$ for all $1 \leqq i \leqq N$, so that the identity is in $P(\mathcal{S})$. $P(\mathcal{S})$ is closed under multiplication, so it is a group. Its sets of imprimitivity are exactly the sets A of the partition Π such that $\mathcal{S} = \bigcup_{A \in \Pi} A \times A$.

Remark 3: If H is a strictly imprimitive group of permutations on $\{1, \cdots, N\}$ with sets of imprimitivity forming the partition Π, and if $\mathcal{S}$ is the smallest stage with $P(\mathcal{S}) \supseteq H$ (Remark 1), then

$$\mathcal{S} = \bigcup_{A \in \Pi} A \times A,$$

so that $\mathcal{S}$ is made of square switches.

Thus, stages of square switches generate strictly imprimitive groups, and any such group can be generated by a stage made of square switches; there is a correspondence between strictly imprimitive groups and stages made of square switches. It has been shown previously[1] that the permutations generated by a stage include a subgroup *only if* the stage contains a substage made of square switches. This suggests that stages made of square switches arise naturally in switching, not just because designers thought of them, but because the mathematics demands it: to factor the symmetric group efficiently into a product of complexes some of which are subgroups, you must use subgroups that are generated by stages of square switches. We have seen that if $\mathcal{S}$ is made of square switches, then $P(\mathcal{S})$ is a strictly imprimitive group; we now show that (*i*) among the stages we would want to consider, those made of square switches are the only ones that generate groups, and (*ii*) only strictly imprimitive groups can be generated by stages.

Theorem 1: *If $P(\mathcal{S})$ contains a group H, then H is strictly imprimitive, and there is a substage $\mathcal{R} \subseteq \mathcal{S} \ni H = P(\mathcal{R})$ and $\mathcal{R}$ is made of square switches*: $\mathcal{R} = \bigcup_{A \in \Pi} A \times A$, *and the sets of imprimitivity of H are just the $A \in \Pi$.*

Proof: Define a relation $\mathcal{R}$ on $\{1, \cdots, N\}$ by the condition that $i\mathcal{R}j$ iff $j = \pi(i)$ for some $\pi \in H$. H must contain the identity, so $i\mathcal{R}i$ holds for all $i \in \{1, \cdots, N\}$. Let i, j, k be numbers in $\{1, \cdots, N\}$, and φ, ψ permutations in H, such that $j = \varphi(i)$ and $k = \psi(j)$. Then $\psi\varphi \in H$ and $k = \psi\varphi(i)$, so that $i\mathcal{R}j$; thus $\mathcal{R}$ is transitive. Finally, if $j = \varphi(i)$ with $\varphi \in H$, we have $i = \varphi^{-1}(j)$ with $\varphi^{-1} \in H$, since H is a group, so $\mathcal{R}$ is symmetric. Thus $\mathcal{R}$ is an equivalence relation, and it is made of square switches. Obviously, $\mathcal{R} \subseteq \mathcal{S}$ and $H = P(\mathcal{R})$, and the result is proved.

To clarify the situation further, we introduce this property of stages:

Definition 7: *$\mathcal{S}$ is complete iff every crosspoint of $\mathcal{S}$ is used in generating some $\pi \in P(\mathcal{S})$, i.e., iff $(i, j) \in \mathcal{S}$ imply $\exists \pi \in P(\mathcal{S}) \ni j = \pi(i)$.*

The point of introducing this idea of completeness is two-fold: (*i*) it gives rise to clean theorems, and (*ii*) it is a reasonable requirement to impose on stages; for it means that every crosspoint can be used to realize some maximal assignment in the stage.

Theorem 2: *$\mathcal{S}$ is made of square switches iff $\mathcal{S}$ is complete and $P(\mathcal{S})$ is a group.*

Proof: If $\mathcal{S}$ is made of square switches, there is a partition Π of $\{1, \cdots, N\}$ such that $\mathcal{S} = \bigcup_{A \in \Pi} A \times A$, and $P(\mathcal{S})$ is clearly the largest strictly imprimitive subgroup of permutations whose sets of imprimitivity are just the $A \in \Pi$. From the form of $\mathcal{S}$ it follows that $(i, i) \in \mathcal{S}$ for $1 \leqq i \leqq N$, so that the identity permutation I belongs to $P(\mathcal{S})$ and $I(i) = i$. Thus, $\mathcal{S}$ is complete. Conversely, let $\mathcal{S}$ be complete and $P(\mathcal{S})$ be a group. We show that $\mathcal{S}$ is an equivalence relation. $P(\mathcal{S})$ must contain the identity, so $(i, i) \in \mathcal{S}$ for each $1 \leqq i \leqq N$, and $\mathcal{S}$ is reflexive. Also, if $(i, j) \in \mathcal{S}$ and $(j, k) \in \mathcal{S}$, then by the completeness there are permutations φ, $\psi \in P(\mathcal{S})$ such that $j = \varphi(i)$ and $k = \psi(j)$. Since $P(\mathcal{S})$ is a group, ψ, $\varphi \in P(\mathcal{S})$ with $\psi\varphi(i) = k$. Hence, $(i, k) \in \mathcal{S}$, and we have shown that $\mathcal{S}$ is transitive. Similarly, if $(i, j) \in \mathcal{S}$ there is a $\varphi \in P(\mathcal{S})$ with $j = \varphi(i)$, whence $i = \varphi^{-1} \in P(\mathcal{S})$, so that $(j, i) \in \mathcal{S}$, and $\mathcal{S}$ is symmetric. It is therefore an equivalence relation, and so is made of square switches.

Remark 4: If $\mathcal{S}$ is reflexive and symmetric, then $\mathcal{S}$ is complete. For given i and j, with $(i, j) \in \mathcal{S}$, we consider the permutation π which

interchanges i and j, leaving all else unchanged. Clearly, $\pi \in P(\mathcal{S})$ because $(j, i) \in \mathcal{S}$, and $(k, k) \in \mathcal{S}$ for all $1 \leqq k \leqq N$ so that $\pi \subseteq \mathcal{S}$.

Remark 5: The condition in Theorem 2 that $\mathcal{S}$ be complete can be replaced by symmetry of $\mathcal{S}$. For if $P(\mathcal{S})$ is a group then $I \in P(\mathcal{S})$, so $\mathcal{S}$ is reflexive. By Remark 4, $\mathcal{S}$ will then be complete.

Let G be a subgroup of S_N.

Theorem 3: $G = P(\mathcal{S})$ *for some stage* $\mathcal{S}$ *iff* G *is strictly imprimitive.*

Proof: $\Leftarrow$ is obvious by Remark 3. Let then $G = P(\mathcal{S})$ for a stage $\mathcal{S}$, and define $\mathcal{R}$ by

$$\mathcal{R} = \{(i, j) : j = \pi(i) \text{ for some } \pi \in G\}.$$

$\mathcal{R}$ is the smallest stage that will generate G. We note that $\mathcal{R}$ is complete, since $(i, j) \in \mathcal{R} \Rightarrow j = \pi(i)$ for some $\pi \in G \subseteq P(\mathcal{R})$. Also $\mathcal{R} \subseteq \mathcal{S}$, because $(i, j) \in \mathcal{R} \Rightarrow \exists \pi \in P(\mathcal{S}) \ni j = \pi(i)$, so that $(i, j) \in \mathcal{S}$. Thus, $P(\mathcal{R}) \subseteq P(\mathcal{S}) = G$, so $P(\mathcal{R})$ is a group. By Theorem 2, $\mathcal{R}$ is made of square switches. Hence, by Remark 2, $G[=P(\mathcal{R})]$ is strictly imprimitive.

VIII. FACTORING OF GROUPS

It is known[1] that the Clos three-stage rearrangeable network (Fig. 3) corresponds to a factorization

$$S_{nr} = G\varphi^{-1}H\varphi G, \tag{2}$$

where $G \sim (S_i)^r$ and $H \sim (S_r)^n$. (For "$\sim$" read "is isomorphic to.") There are similar factorizations of S_N, where N has $p > 2$ prime factors, into a product of $2p - 1$ subgroups associated with a re-

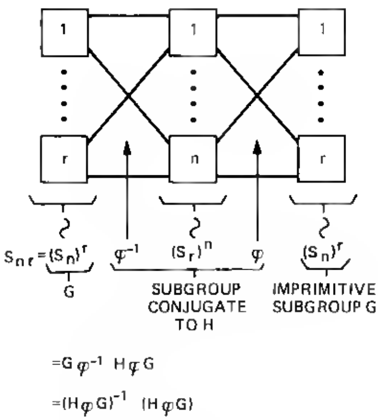

Fig. 3—How the three-stage network factors the group S_{nr}.

arrangeable two-sided network of $2p - 1$ stages symmetrically placed around a center stage. For this reason we shall look at conditions under which a group can be factored into a product of the form

$$H_1H_2 \cdots H_k,$$

where some or all H_i may be subgroups.

For $k = 2$ and H_1, H_2 arbitrary complexes, this problem has been studied by S. Stein[4] under the additional condition that each element of H_1H_2 has a unique representation as a product in H_1H_2. Since, for the study of rearrangeability, this kind of uniqueness is of no interest, we shall not impose Stein's condition while we follow, initially, his original lines of reasoning. For A, B subsets of a group G, we let as usual

$$AB = \{xy : x \in A,\ y \in B\}.$$

Theorem 4: Let G be a group, and A and B subsets of G. Then $G = AB$ iff for every $x \in G$

$$A \cap xB^{-1} \neq \phi.$$

Proof: If $G = AB$, then, given $x \in G$, there exist $a \in A$ and $b \in B$ with $x = ab$, so that $a \in xB^{-1}$. Conversely, if $A \cap xB^{-1}$ is not empty, there exist $a \in A$ and $b \in B$ such that $a = xb^{-1}$, whence $x = ab \in AB$.

If B is a subgroup, the necessary and sufficient condition that $AB = G$ is that $A \cap xB \neq \phi$ for each $x \in G$; for then $B = B^{-1}$. This amounts to saying that A intersects every right coset of B, so by Lagrange's theorem

$$|A| \geqq \frac{|G|}{|B|}.$$

In fact, A can be got by choosing an element from each right coset of B. This is the "best" you can do, given B and no further structure. Of course, analogous results hold for left cosets if A is a group.

The factorization $S_{nr} = G\varphi^{-1}H\varphi G$ corresponding to the three-stage network prompts the question: If G and H are (sub) groups, when is GHG a group? The answer is given in the following:

Theorem 5: Let G, H be groups. Then GHG and HGH are both groups iff they are identical: $GHG = HGH$.

Proof: If GHG is a group, then $(GHG)^2 = GHG$, so that $GHGHG = GHG$. But $HGH \subseteq GHGHG$, so $HGH \subseteq GHG$. Now interchange G and H. Conversely, if $GHG = HGH$, then $(GHG)^2 = GHGHG = GGHGG = GHG$, so GHG is closed and is a group.

The same form of argument actually proves this apparently stronger result:

Theorem 6: *Let G be a group and H a complex. Then GHG is a group iff $HGH \subseteq GHG$.*

These results are extensions to three factors of the familiar fact that if H, G are (sub) groups, then HG is a group iff $GH \subseteq HG$.

Theorem 7: *Let F, G, H be groups. Then FGH is a group iff $GHFG \subseteq FGH$.*

Proof: If FGH is a group it is closed, so $FGHFGH \subseteq FGH$. But if $I \in F \cap H$, then $GHFG \subseteq (FGH)^2$, so that $GHFG \subseteq FGH$. Conversely, should $GHFG$ be contained in FGH, it would follow that $FGHFGH$ was also a subset of FGH. Thus, FGH would be closed and so a group.

A sufficient condition for Theorem 7 to hold is given in the next result, which then allows extension of the sufficiency part of Theorem 6 to three factors.

Lemma 1: *If G is a group, and F and H are complexes such that $HFG \subseteq GHF \subseteq FGH$, then $GHFG \subseteq FGH$.*

Proof: Left-multiplying the first inclusion by G gives at once that $GHFG \subseteq GGHF = GHF \subseteq FGH$.

Theorem 8: *Let F, G, and H be groups. If either $GHF \subseteq FGH \subseteq HFG$ or $HFG \subseteq FGH \subseteq GHF$, then FGH is a group.*

Proof: Assume the first horn of the dilemma. Then by Lemma 1,

$$(FGH)^2 = FGHFGH \subseteq HFG^2H \subseteq FGH,$$

so FGH is closed. Similarly, for the other horn, interchange the roles of GHF and HFG.

Along the same lines, one can give a sufficient condition for the product $G_1G_2\cdots G_n$ of n groups to be a group:

Theorem 9: *If $G_1, G_2, \cdots, G_n$ are all subgroups of a given group, and if*

$$\prod_{i=1}^{n} G_{\pi(i)} = G$$

is the same set for every cyclic permutation $\pi \in S_n$, then $G_1G_2\cdots G_n$ is a group.

$$\begin{aligned} \textit{Proof}: G_1\cdots G_nG_1\cdots G_n &= G_1(G_2\cdots G_{n-1}G_nG_1)G_2\cdots G_n \\ &= (G_1G_2\cdots G_n)G_2\cdots G_n \\ &= (G_3G_4\cdots G_nG_1G_2)G_2\cdots G_n \\ &= (G_1G_2\cdots G_n)G_3\cdots G_n \\ &\ \vdots \\ &= G_1G_2\cdots G_n. \end{aligned}$$

Thus, $\prod_{i=1}^{n} G_i$ *is closed and so is a group.*

It is apparent that the hypothesis of equality of all the sets obtained hy cyclically permuting the order of the multiplication could be replaced hy a continued inclusion as in Theorem 8.

IX. FACTORING BY DOUBLE COSETS: UNIQUENESS

If G and H are (respectively) the groups generated hy a stage of r $n \times n$ switches and a stage of n $r \times r$ switches, and φ is the "canonical" frame cross-connect defined by

$$\varphi: j \rightarrow 1 + \left[\frac{j-1}{n}\right] + r((j-1) \bmod n), \quad j = 1, \cdots, nr, \tag{3}$$

then the Slepian-Duguid theorem affords factorizations

$$S_{nr} = G\varphi^{-1}H\varphi G = H\varphi G\varphi^{-1}H$$

corresponding to Clos rearrangeahle networks. It is a natural pertinent question whether there are *other* cross-connect fields ψ that can be used instead of φ so as to have

$$S_{nr} = G\psi^{-1}H\psi G.$$

We shall show that any such ψ can differ from φ only in having its links mounted in different places on the switches. In particular, and this is the important property, it must give rise (as φ does) to the complete hipartite graph from n nodes to r nodes, when one lets switches he vertices and links he edges in a "frame" corresponding to $G\psi^{-1}H$ (Fig. 4).

First of all, we notice that the Slepian-Duguid theorem can he viewed as a factorization into a product of douhle cosets:

$$S_{nr} = (G\varphi^{-1}H)(H\varphi G) = (G\varphi^{-1}H)(G\varphi^{-1}H)^{-1}.$$

So we are really asking this question: For what double cosets $H\psi G$ is it true that $H\psi G$ times its inverse is the whole group S_{nr}? Note that such douhle cosets are of the form $P(\nu)$ for a conventional frame ν. We make the following convenient definition.

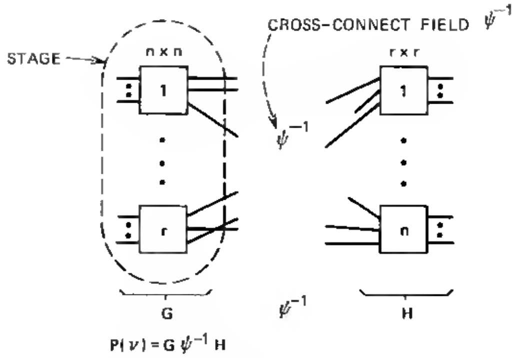

Fig. 4—How a frame generates a double coset.

Definition 8: A permutation ψ works for G and H iff

$$(H\psi G)^{-1}(H\psi G) = S_{nr}.$$

Remark 6: The order of occurrence of groups G and H in Definition 8 is of consequence. For it is readily seen from examples that if ψ works for G and H, it may not work for H and G. However, ψ does work for G and H iff ψ^{-1} works for H and G; for hy Theorem 4, $(H\psi G)^{-1}(H\psi G) = S_{nr}$ iff $x \in S_{nr}$ implies

$$(H\psi G)^{-1} \cap x(H\psi G)^{-1} \neq 0,$$

i.e., iff $x \in S_{nr}$ implies $G\psi^{-1}H \cap xG\psi^{-1}H \neq \phi$. Thus, also, if $G = H$ $(n = r)$, then ψ works for H and H iff ψ^{-1} does too.

The next concept formalizes what is meant by saying that two cross-connect fields differ only in having their links mounted on different terminals of the same switches.

Definition 9: Two permutations (cross-connects) ψ and ξ are equivalent with respect to G and H iff

$$H\psi = \xi G.$$

This amounts to saying that the left ξ-coset of G is exactly the right ψ-coset of H. Intuitively, two cross-connects are equivalent if, when used between columns of switches corresponding to G and H, their links differ only in respect to the terminals on the switches where they attach hut not in the switches themselves; thus, exactly the same pairs of switches have links hetween them, and the cross-connects are in a sense the same except for a renaming of terminals within switches.

What we shall show is that the special, canonical "frame" cross-connect eq. (3) is essentially the only one that works for G and H, in

the sense that any other that does is equivalent to it. Then it follows that there is exactly one double coset $H\psi G$ such that $(H\psi G)^{-1}(H\psi G) = S_{nr}$, namely $H\varphi G$. Thus, there is a unique factorization by double cosets of G and H associated with φ and the complete bipartite graph.

In the next result, G and H are as before the strictly imprimitive subgroups generated respectively by r $n \times n$ switches and n $r \times r$ switches.

Theorem 11: *ψ works for G and H iff ψ is equivalent to φ with respect to G and H, where φ is given by*

$$\varphi: j \rightarrow 1 + [(j-1)/n] + r((j-1) \bmod n), \quad j = 1, \cdots, nr.$$

Proof: If ψ is equivalent to φ with respect to G and H, then the cosets $H\psi G$ and $H\varphi G$ are identical, and so are $(H\psi G)^{-1}$ and $(H\varphi G)^{-1}$. Thus,

$$(H\psi G)^{-1}(H\psi G) = (H\varphi G)^{-1}(H\varphi G),$$

and, thus, ψ works for G and H. That much is fairly obvious. What is interesting is the converse: to prove that we use network arguments. Consider the network ν obtained by placing ψ between a stage of r $n \times n$ switches (giving G) and a stage of n $r \times r$ switches (giving H), followed by ψ^{-1} to another stage of r $n \times n$ switches. If ψ is not equivalent to φ with respect to G and H, then there are switches (in adjacent stages, left and middle, in fact) between which ψ^{-1} places no links, and some between which ψ places two or more links. Let L be a left (or inlet or first stage) switch and M a middle switch with no link between them. There is, then, a right (or outlet or third stage) switch R with one or more links to M. Since all outer switches are square and identical, no assignment taking the inlets of L onto the outlets of R is realizable, because, at most, $n-1$ middle switches have links to both L and R. Thus, the network ν is not rearrangeable, and so some permutation in S_{nr} is missing from $(H\psi G)^{-1}(H\psi G)$. Hence, ψ does not work for G and H.

Remark 7: The argument above shows that if there are permutations ψ and ξ such that $(H\psi G)^{-1}(H\xi G) = S_{nr}$, then ψ and ξ are both equivalent to φ with respect to G and H. Thus, there is a unique factorization of S_{nr} into a product of double cosets of G and H.

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